EXPLICIT DETERMINATION OF IMAGES OF GALOIS REPRESENTATIONS ATTACHED TO HILBERT MODULAR FORMS

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Abstract. In [6] the second author proved that the image of the Galois representation mod λ attached to a Hilbert modular newform is "large" for all but finitely many primes λ, if the form is not a theta series. In this brief note, we give an explicit bound for this exceptional finite set of primes and determine the images in three different examples. Our examples are of Hilbert newforms on real quadratic fields, of parallel or non-parallel weight and of different levels.

1. Introduction

Serre [11] and Ribet [9, 10] proved a large image result for the compatible family of Galois representations attached to a classical modular newform f, provided that f is not a theta series. The result is the following: let E be the number field generated by the eigenvalues of f and for every prime λ in E, let ρ_λ be the continuous λ-adic Galois representation attached to f constructed by Deligne; then for all but finitely many primes λ the image of ρ_λ is large, namely it contains $\mathrm{SL}_2(\mathbb{Z}_\ell)$, with ℓ the rational prime below λ. In fact, the exact value of the image for all but finitely many primes λ can be given if the inner-twists of f (if any) are taken into account. The finite set of primes where the above result fails is usually called the "exceptional set" of the family $\{\rho_\lambda\}$. Explicit versions of the result of Ribet are known and the exceptional sets have been explicitly computed for several examples of classical modular forms [4, 5]. We will now recall a result of [6] that is the natural generalization of Ribet's result to the case of Hilbert modular forms.

Let F be a totally real number field of degree d. Let $k = (k_1, .., k_d)$ be an arithmetic weight (*i.e.* $k_i \geq 2$ are of the same parity) and put $k_0 = \max\{k_i | 1 \leq i \leq d\}$. Let N be an integral ideal of F and ψ a Hecke character of F of conductor dividing N and infinity type $2 - k_0$. We consider a Hilbert modular newform f over F of weight k, level N and central character ψ. By a theorem of Shimura, the Fourier coefficients $c(f, \pi)$ of f (π is a prime of F) generate a number field E. The absolute Galois group of a field L is denoted $\mathcal{G}_L$.

By the work of Ohta, Carayol, Blasius-Rogawski and Taylor [12], for every prime λ of E one can associate to f an absolutely irreducible, totally odd λ-adic representation $\rho_\lambda : \mathcal{G}_F \to \mathrm{GL}_2(E_\lambda)$, unramified outside Nl. By a theorem of Carayol [1] the restriction of ρ_λ to the decomposition group at a prime π of F not dividing ℓ is determined by the local Langlands correspondence for GL_2. In particular, for all primes π of F not dividing Nl we have :

$$\mathrm{tr}(\rho_\lambda(\mathrm{Frob}_\pi)) = c(f, \pi) \qquad \det(\rho_\lambda(\mathrm{Frob}_\pi)) = \psi(\pi)\, \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\pi),$$

where Frob_π denotes a geometric Frobenius at π.

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Denote by $\mathbb{F}_\lambda$ the residue field of E_λ. By taking a Galois stable lattice, we define $\overline{\rho}_\lambda = \rho_\lambda \bmod \lambda : \mathcal{G}_F \to \mathrm{GL}_2(\mathbb{F}_\lambda)$, whose semi-simplification is independent of the particular choice of a lattice.

By [6, Prop.3.8] if f is not a theta series, then for all primes λ outside a finite set of "exceptional primes" the image of $\overline{\rho}_\lambda$ contains $\mathrm{SL}_2(\mathbb{F}_\ell)$.

In this article, we determine explicitly the images and the finite exceptional sets for three examples of Hilbert modular newforms on real quadratic number fields, of different weights and levels. Our results can be summarized in the following table :

f constructed by	F	E	k	N	exceptional λ divide
Consani-Scholten	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{5})$	$(2,4)$	$2\cdot 3\cdot 5$	$2\cdot 3\cdot 5$ (Thm.3.1)
Dembélé	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{6})$	$(2,2)$	$5\cdot(8+3\sqrt{5})$	$2\cdot 3\cdot 5\cdot 19$ (Thm.5.1)
Okada	$\mathbb{Q}(\sqrt{257})$	$\mathbb{Q}(\sqrt{13})$	$(2,2)$	1	$2\cdot 3\cdot 257$ (Thm.6.1)

Section 4 contains a potential application to the modularity of a quintic threefold.

2. Large image results for Hilbert modular forms.

In this section we recall some of the results of [6, §3].

Let v be a prime of F above ℓ of residual degree h. Put $\overline{\rho} = \overline{\rho}_\lambda|_{D_v}$. The semisimplification $\overline{\rho}^{\mathrm{ss}}$ of $\overline{\rho}$ is tamely ramified and the image by $\overline{\rho}^{\mathrm{ss}}$ of the tame inertia I_v^t is cyclic. Let a be a generator of $T = \overline{\rho}^{\mathrm{ss}}(I_v^t)$. Let n be the image of a Frobenuis element by $\overline{\rho}^{\mathrm{ss}}$. Then n belongs to the normalizer N of T in $\mathrm{GL}_2(\mathbb{F}_\lambda)$ and $nan^{-1} = a^{\ell^h}$. Since N/T is of order 2, we have

- either $n \in T$ and $a = a^{\ell^h}$. In this case there exist integers $(p_i)_{0\le i\le 2h-1}$ such that $\overline{\rho}^{\mathrm{ss}}\!: I_v^t \to \mathrm{GL}_2(\mathbb{F}_\lambda)$ factors through the natural map $I_v^t \to \mathbb{F}_{\ell^h}^\times$ followed by

$$F_{\ell^h}^\times \to \mathbb{F}_{\ell^h}^\times \times \mathbb{F}_{\ell^h}^\times \;\;,\;\; x \mapsto (x^{p_0+p_1\ell+\ldots+p_{h-1}\ell^{h-1}}, x^{p_h+p_{h+1}\ell+\ldots+p_{2h-1}\ell^{h-1}}),$$

- either $n \notin T$ and $a = a^{\ell^{2h}}$. In this case there exist integers $(p_i)_{0\le i\le 2h-1}$ such that $\overline{\rho}^{\mathrm{ss}}\!: I_v^t \to \mathrm{GL}_2(\mathbb{F}_\lambda)$ factors through the natural map $I_v^t \to \mathbb{F}_{\ell^{2h}}^\times$ followed by

$$F_{\ell^{2h}}^\times \to \mathbb{F}_{\ell^{2h}}^\times \times \mathbb{F}_{\ell^{2h}}^\times \;\;,\;\; x \mapsto (x^{p_0+p_1\ell+\ldots+p_{2h-1}\ell^{2h-1}}, x^{\ell^h(p_0+p_1\ell+\ldots+p_{2h-1}\ell^{2h-1})}).$$

Proposition 2.1. [6, Cor.2.13] *Assume that $\ell > k_0$ is unramified in F and does not divide N. Then $\overline{\rho}_\lambda$ is crystalline at ℓ, the multisets $\bigcup_{v|\ell}\{p_i | 0 \le i \le 2h-1\}$ and $\{\frac{k_0-k_i}{2}, \frac{k_0+k_i-2}{2} | 1 \le i \le d\}$ are equal and $p_{i+h} = p_i$, for all $0 \le i \le h-1$.*

Using this proposition, one can show :

Proposition 2.2. *Assume that $\ell > k_0$ is unramified in F and does not divide N.*

(i) [6, Prop.3.1(i)] *For all but finitely many primes λ, $\overline{\rho}_\lambda$ is absolutely irreducible.*

(ii) [6, §3.2] *If $d(\ell-1) > 5\sum_{i=1}^d (k_i-1)$ then the image of $\overline{\rho}_\lambda$ in $\mathrm{PGL}_2(\mathbb{F}_\lambda)$ is not isomorphic to one of the groups A_4, S_4 or A_5.*

(iii) [6, Lemma 3.4] *Assume that $\ell \neq 2k_i - 1$ for all $1 \le i \le d$. Assume that the image of $\overline{\rho}_\lambda$ in $\mathrm{PGL}_2(\mathbb{F}_\lambda)$ is dihedral, that is $\overline{\rho}_\lambda \cong \overline{\rho}_\lambda \otimes \varepsilon_{K/F}$ where $\varepsilon_{K/F}$ denotes the character of some quadratic extension K/F. Then K/F is unramified outside N.*

Corollary 2.3. [6, Prop.3.8] *If f is not a theta series then for all but finitely many primes λ the image of $\overline{\rho}_\lambda$ contains (a conjugate of) $\mathrm{SL}_2(\mathbb{F}_\ell)$.*

The proof uses Dickson's classification theorem of the subgroups of $\mathrm{GL}_2(\mathbb{F}_\lambda)$. According to this theorem, the image in $\mathrm{PGL}_2(\mathbb{F}_\lambda)$ of an irreducible subgroup of $\mathrm{GL}_2(\mathbb{F}_\lambda)$ that does not contain (a conjugate of) $\mathrm{SL}_2(\mathbb{F}_\ell)$ is isomorphic either to a dihedral group, either to one of the groups A_4, S_4 or A_5.

3. On an example of Consani-Scholten.

In [2] Consani and Scholten construct a Hilbert modular newform f on $F = \mathbb{Q}(\sqrt{5})$ of weight $(2,4)$, conductor 30 and trivial central character. The Fourier coefficients $c(f,\pi)$ of f belong to $E = \mathbb{Q}(\sqrt{5})$ and are computed for $7 \leq \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\pi) \leq 97$. By Prop.2.1, for all $\ell \geq 7$, $\overline{\rho}_\lambda$ is crystalline and the semi-simplifications of its restrictions to the inertia groups at ℓ satisfy :

- for $\ell = vv'$ split in F :

$$\overline{\rho}_{\lambda|I_v}^{\text{s.s.}} \simeq 1 \oplus \omega^3 \text{ and } \overline{\rho}_{\lambda|I_{v'}}^{\text{s.s.}} \simeq \omega \oplus \omega^2, \text{ or } \overline{\rho}_{\lambda|I_v}^{\text{s.s.}} \simeq \omega_2^3 \oplus \omega_2^{3\ell} \text{ and } \overline{\rho}_{\lambda|I_{v'}}^{\text{s.s.}} \simeq \omega_2^{1+2\ell} \oplus \omega_2^{2+\ell}.$$

- for ℓ inert in F :

$$\overline{\rho}_{\lambda|I_\ell}^{\text{s.s.}} \simeq \omega_2 \oplus \omega_2^{2+3\ell} \text{ or } \omega_2^{1+3\ell} \oplus \omega_2^2 \text{ or } \omega_4^{a+3\ell+(3-a)\ell^2} \oplus \omega_4^{3-a+a\ell^2+3\ell^3}, \text{with } a = 1 \text{ or } 2.$$

Here ω is the cyclotomic character modulo ℓ and ω_2 (resp. ω_4) denotes a fundamental character of level 2 (resp. 4).

The aim of this section is to establish the following

Theorem 3.1. *Let $\ell = \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\lambda) \geq 7$ be a prime. Then the image of $\overline{\rho}_\lambda$ is isomorphic to*
$$\begin{cases} \{\gamma \in \mathrm{GL}_2(\mathbb{F}_\ell), \det(\gamma) \in \mathbb{F}_\ell^{\times 3}\}, \text{ if } \ell \equiv \pm 1 \pmod 5, \\ \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{\ell^2}), \det(\gamma) \in \mathbb{F}_\ell^{\times 3}\}, \text{ if } \ell \equiv \pm 2 \pmod 5. \end{cases}$$

Proof : The theorem would follow from [6, Prop.3.9] once we establish that the image of $\overline{\rho}_\lambda$ is not reducible, nor of dihedral type, nor of A_4, S_4 or A_5 type, nor isomorphic to $\{\gamma \in \mathbb{F}_{\ell^2}^\times \mathrm{GL}_2(\mathbb{F}_\ell)\ ,\ \det(\gamma) \in \mathbb{F}_\ell^{\times 3}\}$ (in the last case we say that $\overline{\rho}_\lambda$ has inner twists).

– Denote by $\epsilon = \frac{1+\sqrt{5}}{2}$ the fundamental unit of F. Then $\epsilon^{120} \equiv 1 \pmod{30}$. By [6, Prop.3.1(ii)] we obtain that $\overline{\rho}_\lambda$ is absolutely irreducible unless $\epsilon^{240} - 1 \in \lambda$ that is (recall that $\ell \geq 7$)

$$\ell \in \{7, 11, 23, 31, 61, 241, 599, 1553, 2161, 20641\} \tag{1}$$

For odd representations irreducibility is equivalent to absolute irreducibility. Since we are only interested in showing the irreducibility for the finite set of primes in (1) it is enough to prove that for each λ in E dividing such a prime ℓ there exists a prime π in F relatively prime to 30ℓ such that the characteristic polynomial of $\bar{\rho}_\lambda(\mathrm{Frob}_\pi)$ is irreducible over $\mathbb{F}_\lambda$. We have used the characteristic polynomials for the primes π in F dividing one of the following primes p:

$$11, 19, 29, 31, 41, 59 \tag{2}$$

Observe that all these primes split in F. For each of these primes π the characteristic polynomial $x^2 - c(f,\pi)x + p^3$ has been computed in [2]. For every prime $\ell \geq 7$ in (1) and for every λ in E dividing ℓ we checked that some of these characteristic polynomials are irreducible modulo λ, and this proves that for every prime $\ell \geq 7$ the residual representations are absolutely irreducible.

– Assume that the image of $\overline{\rho}_\lambda$ is of dihedral type and $\ell \geq 7$. By the local behavior of f at 2 and 3 (*cf* [1]) and by Prop.2.2(iii) there exists a quadratic extension K/F of

discriminant dividing 5 such that for every prime π of F inert in K/F and relatively prime to 30ℓ, $\mathrm{tr}(\overline{\rho}_\lambda(\mathrm{Frob}_\pi)) = 0$. Equivalently, for every such prime π we have $c(f,\pi) \in \lambda$.

Thus, the algorithm for bounding the set of dihedral primes runs as follows: for each quadratic extension K of F unramified outside 5 find a couple of primes π in F verifying π inert in K/F and $c(f,\pi) \neq 0$. The only possibly dihedral primes are those λ dividing these $c(f,\pi)$. Using again the eigenvalues $c(f,\pi)$ for the primes p listed in (2) we have checked with this method that there is no dihedral prime $\ell \geq 7$.

Let us describe in more detail the computations performed : the following list contains all quadratic extensions of $F = \mathbb{Q}(\sqrt{5})$ unramified outside $\sqrt{5}$ (here we use the fact that the multiplicative group of units of F is generated by -1 and $\epsilon = \frac{1+\sqrt{5}}{2}$) :

$$K_1 = F(\sqrt[4]{5});\ K_2 = F\left(\sqrt{\epsilon\sqrt{5}}\right);\ K_3 = F\left(\sqrt{-\sqrt{5}}\right);\ K_4 = F\left(\sqrt{-\epsilon\sqrt{5}}\right)$$

Remark 3.2. It is easy to see that K_4 is the cyclotomic field of 5-th roots of unity.

The primes π in F dividing $p = 29$ or 41 are inert in K_1/F and in K_3/F. The primes π in F dividing $p = 11$ or 29 (resp. $p = 29$ or 59) are inert in K_2/F (resp. in K_4/F). All these primes π verify $c(f,\pi) \neq 0$. In each case, we search for all primes λ dividing $N_p := \mathrm{N}_{E/\mathbb{Q}}\, c(f,\pi)$ for both primes p in the above pairs. By computing the prime factors of each of the N_p :

$$N_{11} : \{2, 11, 19\} \quad N_{29} : \{2, 5, 29\} \quad N_{41} : \{2, 41, 379\} \quad N_{59} : \{2, 5, 59, 71\}$$

we conclude that there are no dihedral primes $\ell \geq 7$.

– By Prop.2.2(ii) the image of $\overline{\rho}_\lambda$ cannot be of A_4, S_4 or A_5 type for $\ell \geq 11$ and a more careful study shows that the argument remains valid for $\ell = 7$.

– Finally, assume that $\overline{\rho}_\lambda$ has inner twists, that is the image of $\overline{\rho}_\lambda$ is isomorphic to $\{\gamma \in \mathbb{F}_{\ell^2}^\times \mathrm{GL}_2(\mathbb{F}_\ell)\ ,\ \det(\gamma) \in \mathbb{F}_\ell^{\times 3}\}$. In this case ℓ should be inert in E and the squares of the traces of the elements of the image of $\overline{\rho}_\lambda$ should belong to $\mathbb{F}_\ell$ (*cf* [6, §3.4]). In particular $c(f,\pi)^2 \in \mathbb{F}_\ell$ for all the primes π in F not dividing 30ℓ. Taking $\pi = (5+\epsilon)$ we find a contradiction. □

4. Towards the modularity of a quintic threefold.

Here we describe a potential application. Consani and Scholten study the middle degree étale cohomology of a quintic threefold $\widetilde{X}$ (a proper and smooth $\mathbb{Z}[\frac{1}{30}]$-scheme with Hodge numbers $h^{3,0} = h^{2,1} = 1$, $h^{2,0} = h^{1,0} = 0$ and $h^{1,1} = 141$). They show that the $\mathcal{G}_\mathbb{Q}$-representation $\mathrm{H}^3(\widetilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is induced from a two dimensional representation σ_λ of $\mathcal{G}_F$, and they conjecture that σ_λ is isomorphic to the ρ_λ from section 3.

Proposition 4.1. *Assume $\ell \geq 7$ and $\overline{\rho}_\lambda = \sigma_\lambda \mod \lambda$. Then σ_λ is modular, and in particular the L-function associated to $\mathrm{H}^3(\widetilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ has an analytic continuation to the whole complex plane and a functional equation.*

Proof : By Thm.3.1 and [6, Prop.3.13] the "large" image condition $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}_\lambda})$ of [6, §0.1] is fulfilled for all $\ell \geq 7$. Then the proposition follows from [7, Thm.0.2]. □

Remark 4.2. (i) In order to prove that $\overline{\rho}_\lambda = \sigma_\lambda \mod \lambda$ it is enough, by the Cebotarev Density Theorem, to check that for "sufficiently many" primes π of F we have $\mathrm{tr}(\sigma_\lambda(\mathrm{Frob}_\pi)) \equiv c(f,\pi) \pmod{\lambda}$. Unfortunately, a relatively small number of these values have been computed.

(ii) If ℓ is inert in E, the image of $\mathrm{Ind}_F^{\mathbb{Q}}\overline{\rho}_\lambda : \mathcal{G}_F \to (\mathrm{GL}_2 \times \mathrm{GL}_2)(\mathbb{F}_{\ell^2})$ is isomorphic to

$$\{(\gamma, \gamma^\sigma) \mid \gamma \in \mathrm{GL}_2(\mathbb{F}_{\ell^2}), \det(\gamma) \in \mathbb{F}_\ell^{\times 3}\}$$

where σ denotes the non-trivial automorphism of $\mathbb{F}_{\ell^2}$. This is a typical example of a representation with non-maximal image satisfying $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}_\lambda})$.

(iii) The results of Taylor[13], if extended to the non-parallel weights k, would give the meromorphic continuation to the whole complex plane and the functional equation.

5. On examples of Dembélé.

In [3, p.60] Dembélé constructs a Hilbert modular newform $f = f_4$ on $F = \mathbb{Q}(\sqrt{5})$ of parallel weight $(2, 2)$, level $N = \sqrt{5}^2(8 + 3\sqrt{5})$, and Fourier coefficients in $E = \mathbb{Q}(\sqrt{6})$.

By Prop.2.1 for all ℓ not dividing $5 \cdot 19$, $\overline{\rho}_\lambda$ is crystalline at ℓ and

$$\overline{\rho}_{\lambda|I_\ell}^{\text{s.s.}} \simeq 1 \oplus \omega \text{ or } \omega_2 \oplus \omega_2^l \text{ or } \omega_4^{1+l} \oplus \omega_4^{\ell^2+\ell^3}.$$

Theorem 5.1. *Assume $\ell \geq 7$ and $\ell \neq 19$. Then the image of $\overline{\rho}_\lambda$ is isomorphic to* $\begin{cases} \mathrm{GL}_2(\mathbb{F}_\ell), \text{ if } \ell \equiv \pm 1, \pm 5 \pmod{24}, \\ \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{\ell^2}), \det(\gamma) \in \mathbb{F}_\ell^\times\}, \text{ otherwise.} \end{cases}$

Proof : As in Thm.3.1, we have to discard the reducible, dihedral, A_4, S_4, A_5 and inner twist cases.

– Assume that $\overline{\rho}_\lambda$ is reducible : $\overline{\rho}_\lambda^{\text{s.s.}} = \varphi \oplus \omega\varphi^{-1}$, where φ is a mod λ crystalline character of $\mathcal{G}_F$. Using the compatibility between local and global Langlands correspondence for GL_2 (*cf* [1]) and the fact that f has trivial central character, we deduce that the prime-to-ℓ part of the conductor of φ divides $\sqrt{5}$. It is easy to check that $\epsilon^4 \equiv 1 \pmod{\sqrt{5}}$ and that λ does not divide $\epsilon^4 - 1$. By the proof of [6, Prop.3.1] it follows that φ is unramified at ℓ, and therefore corresponds (via global class field theory) to a character of the ray class group $\mathrm{Cl}_{F,\sqrt{5}}^+$. Now we use that $\mathrm{Cl}_{F,\sqrt{5}}^+$ is of order 2, because the narrow class group Cl_F^+ is trivial and ϵ^2 generates an index 2 subgroup of $(\mathbb{Z}[\epsilon]/(\sqrt{5}))^\times \cong (\mathbb{F}_5[X]/X^2)^\times$. Hence φ is a trivial or quadratic character. By evaluating $\overline{\rho}_\lambda^{\text{s.s.}} = \varphi \oplus \omega\varphi^{-1}$ at Frob_7 we find a contradiction (the conclusion does not apply to the prime $\lambda = 7$, however in the next few lines we give a second proof of the residual irreducibility that also covers this prime).

Once we know that φ is unramified at ℓ and its conductor divides $\sqrt{5}$, an alternative method consists (without determining the order of φ) of taking primes that are trivial modulo $\sqrt{5}$, more precisely : find primes π in F having a generator α (in F all ideals are principal) such that $\alpha \equiv 1 \pmod{\sqrt{5}}$. It follows from class field theory that for all such primes π, the element Frob_π is totally split in the ray class field of F of conductor $\sqrt{5}$, hence φ is trivial on Frob_π. This property is fulfilled by the two primes of F dividing $p = 11$ and 31. In fact $11 = (-4 + \sqrt{5})(-4 - \sqrt{5})$ and $31 = (6 + \sqrt{5})(6 - \sqrt{5})$. Therefore, equating traces in $\overline{\rho}_\lambda^{\text{s.s.}} = \varphi \oplus \omega\varphi^{-1}$ at these Frob_π we obtain :

$$c(f, -4 + \sqrt{5}) = -2\sqrt{6} \equiv 1 + 11 \pmod{\lambda}, \quad c(f, 6 + \sqrt{5}) = 2 \equiv 1 + 31 \pmod{\lambda}$$

From this, we conclude again that the residual representation is irreducible for $\ell > 5$, $\ell \neq 19$.

– Assume that $\overline{\rho}_\lambda$ is of dihedral type. As in the previous example, by the local behavior of f at $8 + 3\sqrt{5}$ and by Prop.2.2(iii) we have to consider only quadratic extensions of F unramified outside $\sqrt{5}$, that is the same extensions $K_1, ..., K_4$ that we considered in section 3.

For each $1 \leq i \leq 4$, we take again two rational primes p among 11, 29, 41, 59 decomposing as $\pi\pi'$ in F with both π and π' inert in K_i/F. As in the previous example, let us list the prime factors of the norms $N_p = \mathrm{N}_{E/\mathbb{Q}}\, c(f,\pi)$ for each of these p and a suitably chosen $\pi \mid p$:

$$N_{11} : \{2,3\} \qquad N_{29} : \{3,5\} \qquad N_{41} : \{2,3\} \qquad N_{59} : \{3,5\}$$

Since no prime $\ell \geq 7$ divides any of these norms, we conclude that no prime $\ell \geq 7$, $\ell \neq 19$, is dihedral.

– By Prop.2.2(ii) the image of $\overline{\rho}_\lambda$ cannot be of A_4, S_4 or A_5 type for any $\ell \geq 7$.

– As in the proof of Thm.3.1 we see that $\overline{\rho}_\lambda$ cannot have inner twists. □

Remark 5.2. We applied this method to bound explicitly the set of exceptional primes for several other examples of Hilbert modular newforms from Dembélé's tables and it always worked well.

6. On an example of Okada.

Our last example, due to K. Okada [8], concerns a Hilbert modular newform f on $F = \mathbb{Q}(\sqrt{257})$ of weight $(2,2)$, level 1, and Fourier coefficients in $E = \mathbb{Q}(\sqrt{13})$. Here the class number of F is 3.

Theorem 6.1. *Assume $\ell \geq 5$ and $\ell \neq 257$. Then the image of $\overline{\rho}_\lambda$ is isomorphic to*
$$\begin{cases} \mathrm{GL}_2(\mathbb{F}_\ell), \text{ if } \ell \equiv \pm 1, \pm 3, \pm 4 \pmod{13}, \\ \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{\ell^2}), \det(\gamma) \in \mathbb{F}_\ell^\times\}, \text{ otherwise.} \end{cases}$$

Proof : Once again we have to discard the reducible, dihedral, A_4, S_4, A_5 and inner twist cases.

– Assume that $\overline{\rho}_\lambda$ is reducible : $\overline{\rho}_\lambda^{\text{s.s.}} = \varphi \oplus \omega\varphi^{-1}$, where φ is a mod λ crystalline character of $\mathcal{G}_F$. As f is of level 1, the same method as in the previous section shows that φ is everywhere unramified (and therefore of order dividing the class number of F - fact that we won't use).

Assume that $\ell \neq 61$. Let π be one of the two prime ideals of F dividing 61. As π is principal, by class field theory we deduce that φ is trivial on Frob_π. By evaluating $\overline{\rho}_\lambda^{\text{s.s.}} = \varphi \oplus \omega\varphi^{-1}$ at Frob_π and taking traces, we obtain :

$$-1 \pm \sqrt{13} \equiv 1 + 61 = 62 \pmod{\lambda}.$$

When $\ell = 61$, by applying the same argument with a principal prime ideal π of F dividing 67, we get

$$-3 \pm 3\sqrt{13} \equiv 1 + 67 = 68 \pmod{\lambda}.$$

Both these congruences are impossible. Therefore $\overline{\rho}_\lambda$ is irreducible when $\ell \geq 5$ and $\ell \neq 257$.

– As f is of level 1, and F does not have unramified quadratic extensions, it follows from Prop.2.2(iii) that the image of $\overline{\rho}_\lambda$ cannot be of dihedral type for $\ell \geq 5$, $\ell \neq 257$.

– The image of $\overline{\rho}_\lambda$ cannot be of A_4, S_4 or A_5 type for $\ell \geq 7$, $\ell \neq 257$ (*cf* Prop.2.2(ii)). When $\ell = 5$, we observe that the order of the image of $\overline{\rho}_5(\mathrm{Frob}_{17A})$ in $\mathrm{PGL}_2(\mathbb{F}_{25})$ is at least 6.

– As in the proof of Thm.3.1 we see that $\overline{\rho}_\lambda$ cannot have inner twists. □

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